Classification and Recurrence/Transience

Thomas Mountford

EPFL

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MARKOV CHAINS

Definition: For a transition matrix on I, we say that site i *leads to* site j if

$$\exists\ n \geq 0 : P^n_{ij} > 0.$$

We write $i \to j$. Colloquially, we say starting at i we can get to j with strictly positive probability.

Remarks

- Note it is ≥ 0: P^0_{ii} is taken to be 1 so i ALWAYS leads to itself.
- Note that if i leads to j, then this does not mean that a Markov chain with $X_0 = i$ will have $X_n = j$ for some $n \geq 0$.
- Note that if i leads to j, depends on which elements P_{uv} are strictly positive, not at all on the values beyond this.
- i leads to j does not imply that $j$leads to i. E.g. take $P_{uv} \equiv \delta_{iv}$.

Lemma

The following are equivalent for $i, j \in I$

- $i \to j$
- $\exists r \geq 0 : P^r_{ij} > 0 \to j$
- *Either* $i = j$ *or* $\exists r \geq 1 :, i = i_0, i_1 \cdots i_r = j$ *so that* $\forall 1 \leq k \leq r, P_{i_{k-1} i_k} > 0$

Communication

Definition: For a transition matrix on I, we say that sites i and j *communicate* if

$$i \to j \text{ and } j \to i$$

We write $i \leftrightarrow j$.

Lemma

Given a transition matrix P on I, the relation $i \leftrightarrow j$ is an equivalence relation.

The reflexivity and symmetry are immediate, it only remains to choose transitivity. Suppose $i \leftrightarrow j$ and $j \leftrightarrow k$. If j equals i or k then $i \leftrightarrow k$ is immediate. If this is not the case, then for some r and s strictly positive $P_{ij}^r > 0$ and $P_{jk}^s > 0$, $P_{ik}^{r+s} \geq P_{ij}^r P_{jk}^s > 0$, so $i \to k$. Similarly $k \to i$.

Definition: the equivalence classes resulting from this relation are called *communicating classes*.

Irreducibility

Definition: A communicating class C is called *closed* if $i \in C, j \notin C \implies P^r_{ij} = 0 \ \forall r \geq 0$. (Or i does not lead to j). Check: if C is a closed communicating class and $X_0 \in C$ with probability 1, then with probability 1, $X_n \in C \ \forall \ n \geq 0$.
Definition: If $\{i\}$ is a closed communicating class then i is called an *absorbing state*.
] *Definition:* A transition matrix P (or a corresponding Markov chain X is called C is called *irreducible* I is the unique communicating class. If so, then for any initial distribution λ and any $j \in I$, there is a strictly positive chance that for some $n, X_n = j$.

Hitting Probabilities

We have a Markov chain on state space I and two disjoint subsets A and B (with B possibly empty). We address the question

$$\mathbb{P}_i(X \text{ hits } A \text{ before } B)$$

More formally if for $C \subset I$, we define

$$H^C = \inf\{n \geq 0 : X_n \in C\},$$

what is

$$\mathbb{P}_i(H^A < H_B) \equiv h(i).$$

We note that H^C can be zero (if and only if $i \in C$), so the probability is immediate if $x \in A \cup B$. If $i \notin A \cup B$, then by appplying the Markov property at $m = 1$ we obtain

$$h(i) = \sum_j \mathbb{P}_i(H^A < H_B \cap \{X_1 = j\})\mathbb{P}_i(X_1 = j) = \sum_j p_{ij} h(j).$$

So our function h satisfies

a $h(i) \geq 0 \; \forall \; i$

b $h(i) = 1$ for $i \in A$

c $h(i) = 0$ for $i \in B$

d $h(i) = \sum_j p_{ij} h(j)$ for $i \in (A \cup B)^c$

Theorem

The function $h(i) = \mathbb{P}_i(H^A < H_B)$ is the minimal function on I satisfying $a - d$ above.

Proof

We let g be a solution of $a - d$ above. We know by b and c that $g(i) = h(i)$ for $i \in A \cup B$. For other i by d

$$g(i) = \sum_j p_{ij} g(j) = \sum_{j \in A} p_{ij} + \sum_{j \in (A \cup B)^c} p_{ij} g(j) = L_1(i) + R_1(i)$$

$$R_1(i) = \sum_{j \in (A \cup B)^c} p_{ij} \sum_k p_{jk} g(k) = \sum_{j \in (A \cup B)^c} p_{ij} \sum_{k \in A} p_{jk} + \sum_{j \in (A \cup B)^c} p_{ij} \sum_{k \in A \cup B)^c} p_{ij} g(k)$$

$= L_2(i) + R_2(i)$. Continuing, we get

$$g(i) = L_1(i) + L_2(i) \cdots + L_n(i) + R_n(i)$$

where
$L_k(i) = \sum_{j_1 \in (A \cup B)^c} \sum_{j_2 \in (A \cup B)^c} \cdots \sum_{j_{k-1} \in (A \cup B)^c} \sum_{j_k \in A} p_{j_1 j_2} p_{j_2 j_3} \cdots p_{j_{k-1} j_k}$.
$= \mathbb{P}_i(H^A = k, H^B > k)$.

Proof contd

To finish the proof, we simply note that

$$L_1(i) + L_2(i) \cdots + L_n(i) \ \rightarrow \ h(i)$$

as n tends to infinity and that

$$R_n(i) \ \geq \ 0 \ \forall \ n$$

Together these two observations give us that $g(i) \ \geq \ h(i)$.
We get uniqueness if we can guarantee that $\forall i, \ R_n(i)$ tends to zero as n becomes large.

This is not the case in general. E.g. suppose that $\mathbb{P}_i(H^A = H^B = \infty) > 0$ for some $i \in I$, then

$$g(i) \equiv \mathbb{P}_i(H^A < H^B \cup \{H^A = H^B = \infty\})$$

also satisfies $a - d$ and is not equal to h for every i.

Expected time to hit

We now consider $\mathbb{E}(H^A \wedge H^B)$. In fact we drop B and just consider H^A
We define

$$k(i) \;\equiv\; \mathbb{E}_i(H^A)$$

Again if $i \in A$ trivially $k(i) = 0$. As before for $i \notin A$

$$k(i) = \sum_j p_{ij} \mathbb{E}_i(H^A \mid X_1 = j)$$

This time the Markov property gives us that $\mathbb{E}_i(H^A \mid X_1 = j) \;=\; 1 + k(j)$ so we have , for $i \notin A$,

$$k(i) = \sum_j p_{ij}(k(j) + 1) \;=\; 1 + \sum_j p_{ij} k(j)$$

As before k satisfies

- a $k(i) \geq \; 0 \;\forall\; i$
- b $k(i) \;=\; 0$ on A
- c $k(i) = \; 1 \; + \; \sum_j p_{ij} k(j)$ for i not in A.

Expected time to hit

Theorem

The function $k(i) \equiv \mathbb{E}_i(H^A)$ *is the smallest solution satisying* $a - c$ *above.*

Proof.

As before for any solution to $a - c$, g we can write $g(i) = L_0(i) + L_1(i) + L_2(i) \cdots + L_n(i) + R_n(i)$ where $L_0(i) \equiv 1$, $L_n(i) = \mathbb{P}_i(H^A > n)$ and $R_n(i) \geq 0$. So letting n tend to infinity., we obtain

$$g(i) \geq \sum_{n \geq 0} \mathbb{P}_i(H^A > n) = \mathbb{E}_i(H^A) \;=\; k(i)$$

□